



Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Twin Towers of Hanoi

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Available online 28 March 2012

Dedicated to Antonio Machi on the occasion of his retirement

ABSTRACT

In the Twin Towers of Hanoi version of the well known Towers of Hanoi Problem there are two coupled sets of pegs. In each move, one chooses a pair of pegs in one of the sets and performs the only possible legal transfer of a disk between the chosen pegs (the smallest disk from one of the pegs is moved to the other peg), but also, simultaneously, between the corresponding pair of pegs in the coupled set (thus the same sequence of moves is always used in both sets). We provide upper and lower bounds on the length of the optimal solutions to problems of the following type. Given an initial and a final position of n disks in each of the coupled sets, what is the smallest number of moves needed to simultaneously obtain the final position from the initial one in each set? Our analysis is based on the use of a group, called Hanoi Towers group, of rooted ternary tree automorphisms, which models the original problem in such a way that the configurations on n disks are the vertices at level n of the tree and the action of the generators of the group represents the three possible moves between the three pegs. The twin version of the problem is analyzed by considering the action of Hanoi Towers group on pairs of vertices.

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1. Towers of Hanoi and Twin Towers of Hanoi

We first describe the well known Hanoi Towers Problem on n disks and 3 pegs. The n disks have different sizes. Allowed positions (which we call configurations) of the disks on the pegs are those in which no disk is on top of a smaller disk. An example of a configuration on 4 disks is provided in Fig. 1. In a single move, the top disk from one of the pegs can be transferred to the top position on another peg as long as the newly obtained position of the disks is allowed (it is a configuration).

Label the three pegs by 0, 1 and 2. At any moment, regardless of the current configuration, there are exactly three possible moves, denoted by a_{01} , a_{02} , and a_{12} . The move a_{ij} transfers the smallest disk

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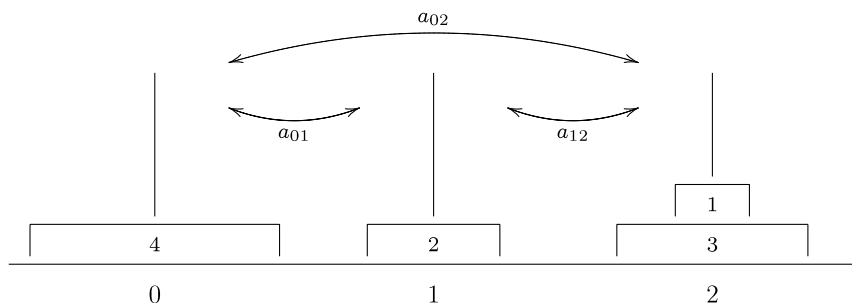


Fig. 1. A configuration on 4 disks.

from pegs i and j between these two pegs. More precisely, if the smallest disk on pegs i and j is on i the move a_{ij} transfers it to j , and if it is on j the move transfers it to i . For instance, the move a_{01} applied to the configuration in Fig. 1 transfers disk 2 from peg 1 to peg 0, a_{02} transfers disk 1 from peg 2 to peg 0, and a_{12} transfers disk 1 from peg 2 to peg 1. We do not need to specify the direction of the transfer, since it is uniquely determined by the disks (by their size) that are currently on pegs i and j . In the exceptional case when there are no disks on either peg i or j , the move a_{ij} leaves such a configuration unchanged.

In the classical Towers of Hanoi Problem on n disks all disks are initially on one of the pegs and the goal is to transfer all of them to another (prescribed) peg in the smallest possible number of moves. It is well known that the optimal solution is unique and consists of $2^n - 1$ moves. One may pose a more general problem such as, given some initial and final configurations on n disks, what is the smallest number of moves needed to obtain the final configuration from the initial one. It turns out that this problem always has a solution (regardless of the chosen initial and final configurations) and that the optimal solution is either unique or there are exactly two solutions. The latter happens for a relatively small number of choices of initial and final configurations. For a survey on topics and results related to Hanoi Towers Problem see [6] and for an optimal solution (represented/obtained by a finite automaton) for any pair of configurations see [8]. Note that, in this setting, none of the instances of the general problem is more difficult (in terms of the optimal number of moves) than the classical problem.

In the Twin Towers of Hanoi version two sets of three pegs labeled by 0, 1 and 2 are coupled up. We often refer to the two sets as the top and the bottom set. A coupled configuration on n disks is a pair of configurations on n disks, one in each set (see, for instance, the coupled configuration on 4 disks in Fig. 2). A move a_{ij} applied to a coupled configuration consists of application of the move a_{ij} to each configuration in the coupled pair. For instance, the move a_{01} applied to the coupled configuration in Fig. 2 transfers disk 1 in the top set to peg 1 and, simultaneously, disk 1 in the bottom set to peg 0. The move a_{02} applied to the same coupled configuration, transfers disk 1 in the top set and disk 2 in the bottom set to peg 2 (in their sets), and a_{12} changes nothing in the top set and transfers disk 1 in the bottom set to peg 2.

In the setting of Twin Towers we pose three problems.

Problem 1 (Twin Towers Switch). Given the initial coupled configuration in which all disks in the top set are on peg 0 and all disks in the bottom set are on peg 2, how many moves are needed to obtain the final coupled configuration in which all disks in the top set are on peg 2, and all disks in the bottom set are on peg 0?

Note that the Twin Towers Switch Problem asks for simultaneous solution of two instances of the classical Hanoi Towers Problem (all disks are, simultaneously, using the same sequence of moves, transferred from peg 0 to peg 2 in the top set, and from peg 2 to peg 0 in the bottom set).

Problem 2 (Small Disk Shift). Given the initial coupled configuration in Fig. 2, how many moves are needed to obtain the final coupled configuration in which all disks are in the same positions as in the initial one, except the smallest disk in each set is moved one peg to the right (disk 1 in the top configuration to peg 1, and disk 1 in the bottom configuration to peg 2)?

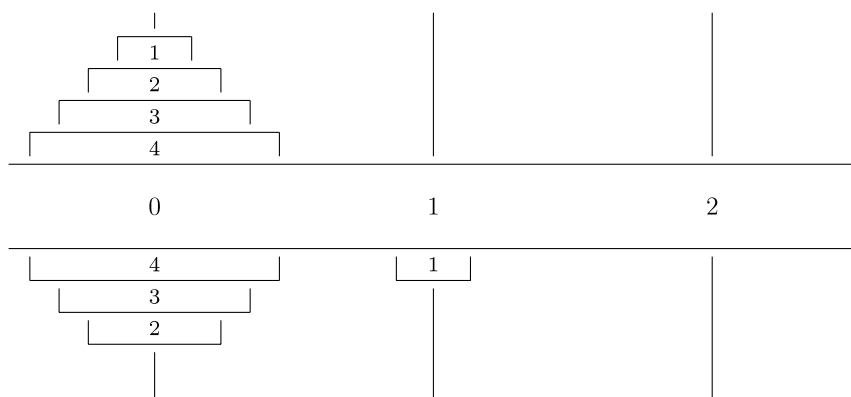


Fig. 2. Initial position for the Small Disk Shift Problem.

Problem 3 (*General Problem*). Given any initial coupled configuration and any final coupled configuration what is the smallest number of moves needed to obtain the final configuration from the initial one?

We provide an upper bound for the Twin Towers Switch, exact answer for the Small Disk Shift, and lower and upper bounds for the General Problem restricted to basic coupled configurations (defined below).

Theorem 1 (*Twin Towers Switch*). The smallest number of moves needed to solve the Twin Towers Switch Problem on n disks is no greater than $a(n)$, where

$$a(n) = \begin{cases} 1, & n = 1, \\ \frac{4}{3} \cdot 2^n - \frac{(-1)^n}{3}, & n \geq 2. \end{cases}$$

Remark 1. The sequence $a(n)$ satisfies the Jacobsthal linear recursion

$$a(n) = a(n-1) + 2a(n-2), \quad \text{for } n \geq 4,$$

with initial condition $a(1) = 1$, $a(2) = 5$, and $a(3) = 11$.

Conjecture 1. The smallest number of moves needed to solve the Twin Towers Switch problem on n disks is exactly $a(n)$.

Note that the Twin Towers Switch, requiring no more than roughly $\frac{4}{3}2^n$ moves is not considerably more difficult than the classical problem of moving a single tower, which requires roughly 2^n moves. In fact, there are more difficult problems that can be posed in the context of coupled sets (recall that there are no problems that are more difficult than the classical problem when only one set of disks is considered). For instance, the next result implies that the Small Disk Shift Problem requires more moves than the Twin Towers Switch Problem.

Theorem 2 (*Small Disk Shift*). The smallest number of moves $d(n)$ needed to solve the Small Disk Shift Problem on n disks is equal to

$$d(n) = \begin{cases} 2, & n = 1, \\ 6, & n = 2, \\ 2 \cdot 2^n, & n \geq 3. \end{cases}$$

In order to state our result on the General Problem, we need the notion of compatible coupled configurations. An initial coupled configuration I on n disks is *compatible* to the final coupled

configuration F on n disks if F can be obtained from I in a finite number of moves. A coupled configuration is called *basic* if the smallest disk in its top configuration and the smallest disk in its bottom configuration are not on corresponding pegs (it is not the case that both are on peg 0, both on peg 1, or both on peg 2).

Note that, based on the branching structure of Hanoi Towers group described by Grigorchuk and the author in [3], D'Angeli and Donno show in [1] that Hanoi Towers group acts distance 2-transitively on the levels of the rooted ternary tree. This provides a characterization of the pairs of compatible coupled configurations. In particular, their result implies that all basic coupled configurations are compatible. We quote their result in more detail (Theorem 6), after we sufficiently develop the necessary terminology. Along the way we provide a different proof (we need it for our upper bound estimate on the General Problem). Note that an interesting consequence of the result of D'Angeli and Donno is that the Hanoi Towers group induces an infinite sequence of finite Gel'fand pairs (see [1] for details).

Theorem 3 (General Problem for Basic Configurations). *The number of moves needed to obtain one basic coupled configuration on n disks from another is no greater than*

$$\frac{11}{3} \times 2^n = 3.\overline{66} \times 2^n.$$

Note that the coupled configurations in Theorem 2 are basic. Thus, Theorem 2 implies that for at least one pair of basic coupled configurations the smallest number of moves that is needed to obtain one from the other is exactly 2×2^n .

Obtaining good upper bound seems to be a difficult task, since one needs to solve all instances of the problem in optimal or nearly optimal way. Lower bounds seem a bit easier to obtain since they may be derived from lower bounds from some specific, well chosen, instances. The lower bound (2×2^n) and the upper bound (3.66×2^n) provided here differ by less than a factor of two.

All results mentioned so far will be recast in the following sections in the natural setting of group actions on rooted trees (equivalently, prefix and length preserving actions on finite words over a finite alphabet). The reason is that this setting provides a convenient language and tools to prove our results.

2. Encoding by words and tree automorphisms

We start by an encoding of the original Hanoi Towers Problem on three pegs, as originally presented in [5] (and further elaborated in [3,4]), by a group of rooted ternary tree automorphisms.

Label the disks by $1, 2, \dots, n$ according to their size (smallest to largest). The configurations can be encoded by words over the finite alphabet $X = \{0, 1, 2\}$. The letters in this alphabet represent the pegs. The word $x_1 x_2 \dots x_n$ represents the unique configuration on n disks in which, for $i = 1, \dots, n$, the disk i is on peg x_i . For example, the word 2120 represents the configuration in Fig. 1. Note that there are exactly 3^n configurations on n disks.

The moves a_{ij} are encoded as the transformations of the set of all finite words X^* over X defined by

$$\begin{aligned} a_{01}(2 \dots 20u) &= 2 \dots 21u, & a_{02}(1 \dots 10u) &= 1 \dots 12u, & a_{12}(0 \dots 01u) &= 0 \dots 02u, \\ a_{01}(2 \dots 21u) &= 2 \dots 20u, & a_{02}(1 \dots 12u) &= 1 \dots 10u, & a_{12}(0 \dots 02u) &= 0 \dots 01u, \\ a_{01}(2 \dots 2) &= 2 \dots 2, & a_{02}(1 \dots 1) &= 1 \dots 1, & a_{12}(0 \dots 0) &= 0 \dots 0, \end{aligned}$$

for any word u in X^* . Thus, a_{ij} changes the first occurrence of i or j to the other of these two symbols. The point of, say, a_{01} "ignoring" initial prefixes of the form 2^k is that such prefixes represent small disks on peg 2, and a_{01} should ignore such disks, since it is supposed to transfer a disk between peg 0 and peg 1. The first occurrence of 0 or 1 represents the smallest disk on one of these two pegs and changing this occurrence of the symbol 0 or 1 to the other one in the code of the given configuration transfers the corresponding disk to the other peg. Note that if a_{ij} is applied to (a code of) a configuration that has no occurrences of i or j it leaves such a configuration unchanged. This corresponds to the situation in which there are no disks on pegs i and j and the move a_{ij} has no effect on such a configuration since there are no disks to be moved.

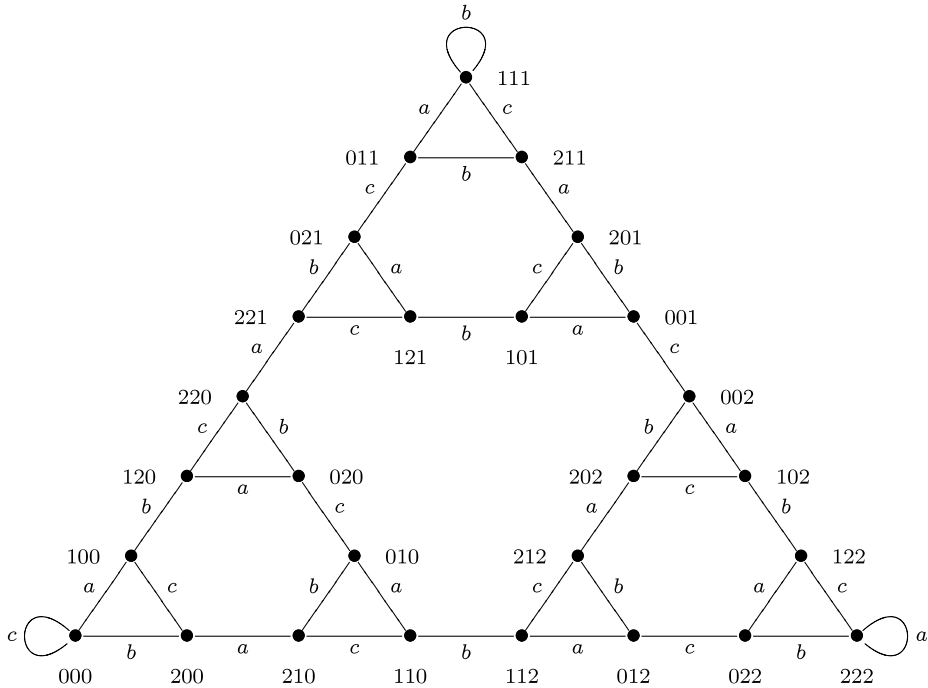


Fig. 3. Γ_3 , the Hanoi graph on 3 disks.

In order to work with more compact notation, set

$$a_{01} = a, \quad a_{02} = b, \quad a_{12} = c.$$

In this notation, the moves a , b and c act on the set of all finite words X^* by

$$\begin{aligned} a(2 \cdots 20u) &= 2 \cdots 21u, & b(1 \cdots 10u) &= 1 \cdots 12u, & c(0 \cdots 01u) &= 0 \cdots 02u, \\ a(2 \cdots 21u) &= 2 \cdots 20u, & b(1 \cdots 12u) &= 1 \cdots 10u, & c(0 \cdots 02u) &= 0 \cdots 01u, \\ a(2 \cdots 2) &= 2 \cdots 2, & b(1 \cdots 1) &= 1 \cdots 1, & c(0 \cdots 0) &= 0 \cdots 0. \end{aligned} \quad (1)$$

The Hanoi graph on n disks, denoted by Γ_n , is the graph on 3^n vertices representing the configurations on n disks. Two vertices u and v are connected by an edge labeled by $s \in \{a, b, c\}$ if the configurations represented by u and v can be obtained from each other by application of the move s (note that each of the moves is an involution). The Hanoi graph on 3 disks is depicted in Fig. 3. Graphs very similar to the graphs we just defined have already appeared in the literature in connection to Hanoi Towers Problem (see, for instance, [6]). The difference is that the edges are usually not labeled and there are no loops at the corners.

The set of all words X^* has the structure of a rooted ternary tree in which the root is the empty word, level n of the tree consists of the 3^n words of length n over X , and each vertex (each word) u has three children, $u0$, $u1$ and $u2$. The transformations a , b and c act on the tree X^* as tree automorphisms (in particular, they preserve the root and the levels of the tree). Thus, a , b and c generate a group of automorphisms of the rooted ternary tree X^* . The group $H = \langle a, b, c \rangle$, called Hanoi Towers group, was defined in [5]. The Hanoi graph Γ_n is the Schreier graph, with respect to the generating set $\{a, b, c\}$, of the action of H on the words of length n in X^* (Schreier graph of the action on level n in the tree).

A sequence of moves is a word over $S = \{a, b, c\}$. The order in which moves are applied is from right to left as in the following calculation

$$caba(00220) = cab(1220) = ca(1020) = c(0020) = 0010.$$

The structure of the Hanoi graphs is fairly well understood. In particular, for $n \geq 0$, the Hanoi graph Γ_{n+1} is obtained from the Hanoi graph Γ_n as follows [3]. Three copies of Γ_n are constructed by appending the label 0, 1, and 2, respectively, to every vertex label in Γ_n . Then the two loops labeled by c at the vertices $0^n 1$ and $0^n 2$ are deleted and replaced by an edge between $0^n 1$ and $0^n 2$ labeled by c , the two loops labeled by b at the vertices $1^n 0$ and $1^n 2$ are deleted and replaced by an edge between $1^n 0$ and $1^n 2$ labeled by b , and the two loops labeled by a at the vertices $2^n 0$ and $2^n 1$ are deleted and replaced by an edge between $2^n 0$ and $2^n 1$ labeled by a . Indeed, this “rewiring” on the next level (level $n + 1$) needs to be done as indicated since $c(0^n 1) = 0^n 2$, $b(1^n 0) = 1^n 2$ and $a(2^n 0) = 2^n 1$. In general, the graphs for even and odd n have the form provided in Figs. 4 and 5. These figures suffice for our purposes, since only the region near the path from 0^n to 2^n (near the bottom) and near the path from 0^n to 1^n (near the left side) play significant role in our considerations.

The following lemma, providing a non-recursive, optimal solution to the classical Hanoi Towers Problem is part of the folklore (it has been proved and expressed in many disguises and our setting may be considered one of them).

Lemma 1. *The diameter of the Hanoi Towers graph Γ_n is $2^n - 1$. It is achieved as the distance between any two of the configurations 0^n , 1^n , and 2^n . The unique sequence of moves of length $2^n - 1$ between any two of these configurations is given in the following table.*

From \ To	Even n			Odd n		
	0^n	1^n	2^n	0^n	1^n	2^n
0^n	\times	$(cab)^{m(n)}$	$(cba)^{m(n)}$	\times	$a(cba)^{m(n)}$	$b(cab)^{m(n)}$
1^n	$(bac)^{m(n)}$	\times	$(bca)^{m(n)}$	$a(bca)^{m(n)}$	\times	$c(bac)^{m(n)}$
2^n	$(abc)^{m(n)}$	$(acb)^{m(n)}$	\times	$b(acb)^{m(n)}$	$c(abc)^{m(n)}$	\times

where $m(n) = \frac{1}{3}(2^n - 1)$, for even n , and $m(n) = \frac{1}{3}(2^n - 2)$, for odd n .

Our goal is to provide some understanding of the coupled Hanoi graph $C\Gamma_n$ on n disks. The vertices of this graph are the 3^{2n} pairs of words $\begin{pmatrix} u_T \\ u_B \end{pmatrix}$ of length n over X (representing the top and the bottom configuration on n disks in a coupled configuration). Two vertices in $C\Gamma_n$ are connected by an edge labeled by s in $\{a, b, c\}$ if the coupled configurations represented by these vertices can be obtained from each other by application of the move s . The coupled Hanoi graph on 1 disk is depicted in Fig. 6. The coupled Hanoi graph $C\Gamma_n$ is the Schreier graph, with respect to the generating set $\{a, b, c\}$, of the action of H on the pairs of words of length n in X^* defined by

$$s \begin{pmatrix} u_T \\ u_B \end{pmatrix} = \begin{pmatrix} s(u_T) \\ s(u_B) \end{pmatrix},$$

for s in $\{a, b, c\}$.

3. Twin Towers Switch

In this section we provide an upper bound on the number of moves needed to solve the Twin Towers Switch Problem. In the language of coupled Hanoi graphs the same result is expressed as follows.

Theorem 4 (Twin Towers Switch). *The distance between the coupled configurations*

$$\begin{pmatrix} 000 \cdots 0 \\ 222 \cdots 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 222 \cdots 2 \\ 000 \cdots 0 \end{pmatrix}$$

in the coupled Hanoi graph $C\Gamma_n$ (on n disks) is no greater than

$$a(n) = \begin{cases} 1, & n = 1, \\ \frac{4}{3} \cdot 2^n - \frac{(-1)^n}{3}, & n \geq 2. \end{cases}$$

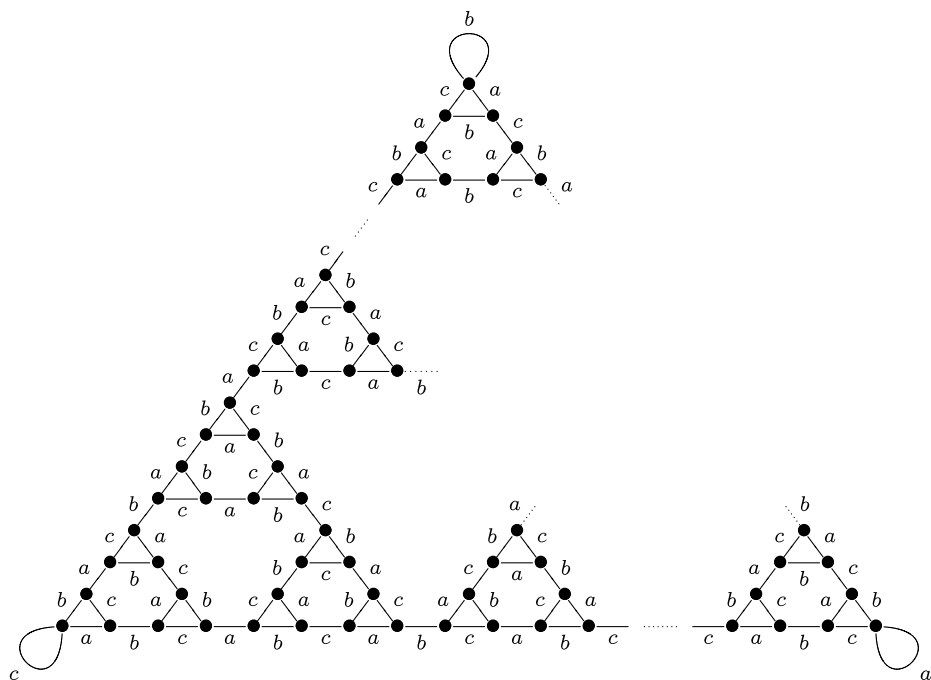


Fig. 4. Hanoi graph on even number of disks.

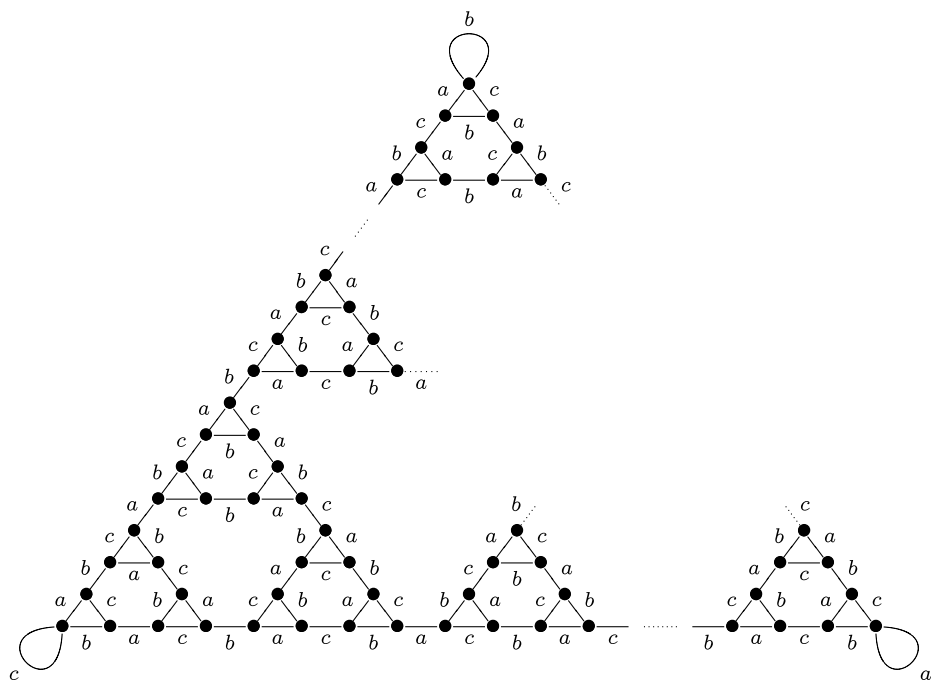
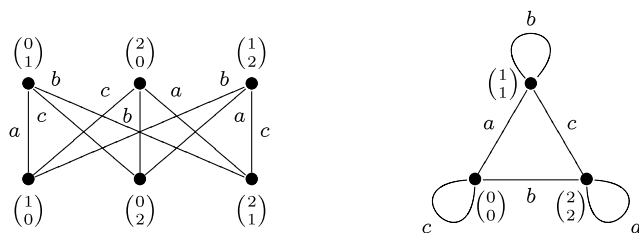


Fig. 5. Hanoi graph on odd number of disks.

Fig. 6. CG_1 , the coupled Hanoi graph on 1 disk.

Proof. Let $n = 2$. We have, by using (1),

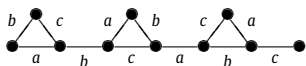
$$ababa(00) = abab(10) = aba(12) = ab(02) = a(22) = 22.$$

Since $ababa$ is a palindrome, it has order 2 (as a group element) and, therefore, $ababa(22) = 00$. Thus the distance between the initial and the final coupled configurations is no greater than 5 (it can be shown that it is actually 5).

Assume that n is even and $n \geq 4$. Consider the sequence of $a(n) = \frac{4}{3} \cdot 2^n - \frac{1}{3}$ moves

$$ababa(cacababa)^{\frac{1}{3}(2^{n-1}-2)}.$$

Notice the pattern



that repeats along the bottom edge in Fig. 4, indicating that the result of the action of $cacababa$ and $(cba)^2$ on the leftmost vertex in the pattern is the same and it is equal to the rightmost vertex in the pattern (which is the leftmost vertex in the next occurrence of the pattern). Therefore, by (1) and Lemma 1,

$$\begin{aligned} ababa(cacababa)^{\frac{1}{3}(2^{n-1}-2)}(000 \dots 0) &= ababa(cba)^{\frac{1}{3}(2^{n-4})}(000 \dots 0) \\ &= ababaabc(cba)^{\frac{1}{3}(2^{n-1})}(000 \dots 0) \\ &= abac(222 \dots 2) \\ &= aba(122 \dots 2) = ab(022 \dots 2) \\ &= a(222 \dots 2) = 222 \dots 2. \end{aligned}$$

Since $ababa(cacababa)^{\frac{1}{3}(2^{n-1}-2)}$ is a palindrome it has order 2. Thus

$$ababa(cacababa)^{\frac{1}{3}(2^{n-1}-2)}(222 \dots 2) = 000 \dots 0$$

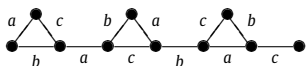
and the distance between the initial and the final coupled configurations is no greater than $a(n)$.

Let $n = 1$. The distance between the coupled configurations $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is 1 (see Fig. 6).

Assume that n is odd and $n \geq 3$. Consider the sequence of $a(n) = \frac{4}{3} \cdot 2^n + \frac{1}{3}$ moves

$$aca(cbc bcaca)^{\frac{1}{3}(2^{n-1}-1)}.$$

Notice the pattern



that repeats along the bottom edge in Fig. 5, indicating that the result of the action of $cbcb caca$ and $(cab)^2$ on the leftmost vertex in the pattern is the same and it is equal to the rightmost vertex in the

pattern (which is the leftmost vertex in the next occurrence of the pattern). Therefore, by (1) and Lemma 1,

$$\begin{aligned}aca(cbc bcaca)^{\frac{1}{3}(2^{n-1}-1)}(000 \dots 0) &= aca(cab)^{\frac{1}{3}(2^n-2)}(000 \dots 0) \\&= acabb(cab)^{\frac{1}{3}(2^n-2)}(000 \dots 0) = acab(222 \dots 2) \\&= aca(022 \dots 2) = ac(122 \dots 2) \\&= a(222 \dots 2) = 222 \dots 2.\end{aligned}$$

Since $aca(cbc bcaca)^{\frac{1}{3}(2^{n-1}-1)}$ is a palindrome it has order 2. Thus

$$aca(cbc bcaca)^{\frac{1}{3}(2^{n-1}-1)}(222 \dots 2) = 000 \dots 0$$

and the distance between the initial and the final coupled configurations is no greater than $a(n)$. \square

Remark 2. There are several solutions of length $a(n)$, for $n \geq 2$. For instance, another solution, for odd n , is

$$cac(ababacac)^{\frac{1}{3}(2^{n-1}-1)}$$

and, for even n , is

$$cbcbc(acacbc bc)^{\frac{1}{3}(2^{n-1}-2)}.$$

We rephrase Conjecture 1 as follows.

Conjecture 2. The distance between the coupled configurations

$$\begin{pmatrix} 000 \dots 0 \\ 222 \dots 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 222 \dots 2 \\ 000 \dots 0 \end{pmatrix}$$

in the coupled Hanoi graph CG_n (on n disks) is equal to $a(n)$.

4. Small Disk Shift

For the considerations that follow, the concept of parity will be useful.

Definition 1. For a configuration $u = x_1 \dots x_n$ in X^* and $x \in X$, let $p_x(u)$ be the parity of the number of appearances of the letter x in u . For a coupled configuration $U = \begin{pmatrix} u_T \\ u_B \end{pmatrix}$, let $p_x(U)$ be the parity of the sum of $p_x(u_T)$ and $p_x(u_B)$.

Call any of the configurations 0^n , 1^n , 2^n a *corner configuration*. Call a coupled configuration a *corner coupled configuration* if at least one of the configurations in it is a corner configuration. Application of a_{ij} to any non corner configuration changes the parities of both i and j . Therefore, application of a_{ij} to a non corner coupled configuration does not change any parities.

Theorem 5 (Small Disk Shift). The distance between the coupled configurations

$$\begin{pmatrix} 000 \dots 0 \\ 100 \dots 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 100 \dots 0 \\ 200 \dots 0 \end{pmatrix}$$

in the coupled Hanoi graph CG_n (on n disks) is

$$d(n) = \begin{cases} 2, & n = 1, \\ 6, & n = 2, \\ 2 \cdot 2^n, & n \geq 3. \end{cases}$$

Proof (Proof of Theorem 5: Upper Bound). Assume that n is even and $n \geq 4$. Consider the sequence of $2 \cdot 2^n$ moves

$$bab(abc)^{\frac{1}{3}(2^n-4)}a(cba)^{\frac{1}{3}(2^n-1)}c.$$

A simple and convincing way to verify that this sequence of moves accomplishes the goal would be to trace the action in Fig. 4. A more formal approach, using (1) and Lemma 1, gives

$$\begin{aligned} & bab(abc)^{\frac{1}{3}(2^n-4)}a(cba)^{\frac{1}{3}(2^n-1)}c(000 \dots 0) \\ &= bab(abc)^{\frac{1}{3}(2^n-4)}a(cba)^{\frac{1}{3}(2^n-1)}(000 \dots 0) \\ &= bab(abc)^{\frac{1}{3}(2^n-4)}a(222 \dots 2) = bab(abc)^{\frac{1}{3}(2^n-4)}(222 \dots 2) \\ &= babcba(abc)^{\frac{1}{3}(2^n-1)}(222 \dots 2) = babcba(000 \dots 0) \\ &= babcb(100 \dots 0) = babc(120 \dots 0) \\ &= bab(220 \dots 0) = ba(020 \dots 0) \\ &= b(120 \dots 0) = 100 \dots 0, \end{aligned}$$

and

$$\begin{aligned} & bab(abc)^{\frac{1}{3}(2^n-4)}a(cba)^{\frac{1}{3}(2^n-1)}c(100 \dots 0) \\ &= bab(abc)^{\frac{1}{3}(2^n-4)}a(cba)^{\frac{1}{3}(2^n-1)}(200 \dots 0) \\ &= bab(abc)^{\frac{1}{3}(2^n-4)}acba(200 \dots 0) = bab(abc)^{\frac{1}{3}(2^n-4)}acb(210 \dots 0) \\ &= bab(abc)^{\frac{1}{3}(2^n-4)}ac(010 \dots 0) = bab(abc)^{\frac{1}{3}(2^n-4)}a(020 \dots 0) \\ &= bab(abc)^{\frac{1}{3}(2^n-4)}(120 \dots 0) = bab(120 \dots 0) \\ &= ba(100 \dots 0) = b(000 \dots 0) \\ &= 200 \dots 0, \end{aligned}$$

where, in the transition between the first and second row, we used the fact that $\frac{1}{3}(2^n - 1)$ is odd.

Assume that n is odd and $n \geq 3$. Consider the sequence of $2 \cdot 2^n$ moves

$$bab(abc)^{\frac{1}{3}(2^n-5)}bcba(cba)^{\frac{1}{3}(2^n-2)}.$$

A simple and convincing way to verify that this sequence of moves accomplishes the goal would be to trace the action in Fig. 5. A more formal approach, using (1) and Lemma 1, gives

$$\begin{aligned} & bab(abc)^{\frac{1}{3}(2^n-5)}bcba(cba)^{\frac{1}{3}(2^n-2)}(000 \dots 0) \\ &= bab(abc)^{\frac{1}{3}(2^n-5)}bcb(111 \dots 1) \\ &= bab(abc)^{\frac{1}{3}(2^n-5)}aabc(111 \dots 1) = baba(bca)^{\frac{1}{3}(2^n-5)}abc(111 \dots 1) \\ &= baba(bca)^{\frac{1}{3}(2^n-5)}ab(211 \dots 1) = baba(bca)^{\frac{1}{3}(2^n-5)}a(011 \dots 1) \\ &= baba(bca)^{\frac{1}{3}(2^n-5)}(111 \dots 1) = babcbaabca(bca)^{\frac{1}{3}(2^n-5)}(111 \dots 1) \\ &= babcbaa(bca)^{\frac{1}{3}(2^n-2)}(111 \dots 1) = babcba(000 \dots 0) \\ &= babcb(100 \dots 0) = babc(120 \dots 0) \\ &= bab(220 \dots 0) = ba(020 \dots 0) \\ &= b(120 \dots 0) = 100 \dots 0, \end{aligned}$$

and

$$\begin{aligned} & bab(abc)^{\frac{1}{3}(2^n-5)}bcba(cba)^{\frac{1}{3}(2^n-2)}(100 \dots 0) \\ &= bab(abc)^{\frac{1}{3}(2^n-5)}b(100 \dots 0) \end{aligned}$$

$$\begin{aligned}
&= bab(abc)^{\frac{1}{3}(2^n-5)}(120 \dots 0) = bab(120 \dots 0) \\
&= ba(100 \dots 0) = b(000 \dots 0) \\
&= 2000 \dots 0.
\end{aligned}$$

When $n = 1$, a solution of length 2 is given by the sequence of moves ba and, for $n = 2$, a solution of length 6 is given by the sequence of moves $bcacba$. \square

Remark 3. Note that the above sequences of moves of length $2 \cdot 2^n$ are not unique. For instance, for even n , $n \geq 4$, one could use

$$caba(bac)^{\frac{1}{3}(2^n-4)}b(cab)^{\frac{1}{3}(2^n-1)}.$$

Proof (Proof of Theorem 5: Lower Bound). Since the 0-parities for the initial and final coupled configurations are

$$p_0 \begin{pmatrix} 000 \dots 0 \\ 100 \dots 0 \end{pmatrix} = 1 \quad \text{and} \quad p_0 \begin{pmatrix} 100 \dots 0 \\ 200 \dots 0 \end{pmatrix} = 0,$$

somewhere on the way from the initial to the final coupled configuration the 0-parity changes. This parity cannot be changed at the corner coupled configurations $\begin{pmatrix} 0^n \\ v \end{pmatrix}$ and $\begin{pmatrix} v \\ 0^n \end{pmatrix}$, where v is not a corner configuration. Since the 0-parity must be changed, any sequence of moves that starts at the initial coupled configuration $\begin{pmatrix} 000 \dots 0 \\ 100 \dots 0 \end{pmatrix}$ and accomplishes this change involves a corner a -loop or a corner b -loop application in either the top or in the bottom configuration. The 4 possibilities are given as cases Top_a , Top_b , Bot_a and Bot_b (standing for top configuration involved in a corner a -loop, top configuration involved in a corner b -loop, etc.) in Table 1, where, in each case, $*$ denotes some configuration different from the one with which it is coupled. The last two columns provide the number of steps in the unique shortest path of the given form, for even and odd number of disks.

Note that the above considerations already show that $d(n) \geq 2 \cdot 2^n - 2$ and that the largest disk has to be moved in at least one coupled set of disks.

Further, any element g in H for which $g \begin{pmatrix} 000 \dots 0 \\ 100 \dots 0 \end{pmatrix} = \begin{pmatrix} 100 \dots 0 \\ 200 \dots 0 \end{pmatrix}$ must act on the first letter as the permutation (012), which is an even permutation. Therefore, the length of g must be even. To complete the proof, all we need to show is that none of the shortest paths (sequences of moves) of length $2 \cdot 2^n - 2$ implicitly mentioned in Table 1 solves the Small Disk Shift Problem.

For the unique shortest path g of length $2 \cdot 2^n - 2$ in Case Top_a , even n , such that for the top configuration we have $g(000 \dots 0) = 100 \dots 0$, tracing the action in Fig. 4 for the bottom configuration, we obtain

$$(bca)^{\frac{1}{3}(2^n-1)}(cba)^{\frac{1}{3}(2^n-1)}(100 \dots 0) = 201 \dots 1 \neq 200 \dots 0.$$

For the unique shortest path g of length $2 \cdot 2^n - 2$ in Case Bot_a , even n , such that for the bottom configuration we have $g(100 \dots 0) = 200 \dots 0$, tracing the action in Fig. 4 for the top configuration, we obtain

$$(cbc)(abc)^{\frac{1}{3}(2^n-4)}(acb)^{\frac{1}{3}(2^n-1)}(000 \dots 0) = 101 \dots 1 \neq 100 \dots 0.$$

For the unique shortest path g of length $2 \cdot 2^n - 2$ in Case Bot_b , even n , such that for the bottom configuration we have $g(100 \dots 0) = 200 \dots 0$, tracing the action in Fig. 4 for the top configuration, we obtain

$$ac(bac)^{\frac{1}{3}(2^n-4)}(bca)^{\frac{1}{3}(2^n-1)}c(000 \dots 0) = 102 \dots 2 \neq 100 \dots 0.$$

For the unique shortest path g of length $2 \cdot 2^n - 2$ in Case Top_b , odd n , such that for the top configuration we have $g(000 \dots 0) = 100 \dots 0$, tracing the action in Fig. 5 for the bottom configuration, we obtain

$$(bca)^{\frac{1}{3}(2^n-2)}ba(cba)^{\frac{1}{3}(2^n-2)}(100 \dots 0) = 222 \dots 2 \neq 200 \dots 0.$$

Table 1
Changing the 0-parity.

Case	Initial	0-parity Change	Even n	Odd n
Top _a :	$\begin{pmatrix} 00\cdots 0 \\ 10\cdots 0 \end{pmatrix}$	$\xrightarrow{a} \begin{pmatrix} 2^n \\ * \end{pmatrix} \rightarrow \begin{pmatrix} 10\cdots 0 \\ * \end{pmatrix}$	$2 \cdot 2^n - 2$	$2 \cdot 2^n - 1$
Top _b :	$\begin{pmatrix} 00\cdots 0 \\ 10\cdots 0 \end{pmatrix}$	$\xrightarrow{b} \begin{pmatrix} 1^n \\ * \end{pmatrix} \rightarrow \begin{pmatrix} 10\cdots 0 \\ * \end{pmatrix}$	$2 \cdot 2^n - 1$	$2 \cdot 2^n - 2$
Bot _a :	$\begin{pmatrix} 00\cdots 0 \\ 10\cdots 0 \end{pmatrix}$	$\xrightarrow{a} \begin{pmatrix} * \\ 2^n \end{pmatrix} \rightarrow \begin{pmatrix} * \\ 20\cdots 0 \end{pmatrix}$	$2 \cdot 2^n - 2$	$2 \cdot 2^n - 2$
Bot _b :	$\begin{pmatrix} 00\cdots 0 \\ 10\cdots 0 \end{pmatrix}$	$\xrightarrow{b} \begin{pmatrix} * \\ 1^n \end{pmatrix} \rightarrow \begin{pmatrix} * \\ 20\cdots 0 \end{pmatrix}$	$2 \cdot 2^n - 2$	$2 \cdot 2^n - 2$

For the unique shortest path g of length $2 \cdot 2^n - 2$ in Case Bot_a, odd n , such that for the bottom configuration we have $g(100 \cdots 0) = 200 \cdots 0$, tracing the action in Fig. 5 for the top configuration, we obtain

$$(acb)^{\frac{1}{3}(2^n-2)} a(bca)^{\frac{1}{3}(2^n-2)} c(000 \cdots 0) = 111 \cdots 1 \neq 100 \cdots 0.$$

Finally, for the unique shortest path g of length $2 \cdot 2^n - 2$ in Case Bot_b, odd n , such that for the bottom configuration we have $g(100 \cdots 0) = 200 \cdots 0$, tracing the action in Fig. 5 for the top configuration, we obtain

$$c(bca)^{\frac{1}{3}(2^n-2)} b(acb)^{\frac{1}{3}(2^n-2)} (000 \cdots 0) = 122 \cdots 2 \neq 100 \cdots 0. \quad \square$$

5. General Problem

In this section we describe the compatible coupled configurations (recovering the result of D'Angeli and Donno from [1]) and then provide an upper bound on the distance between any compatible coupled configurations.

In order to accomplish the goals of this section, we need a bit more information on the Hanoi Towers group H . In particular, we rely on the self-similarity of the action of H on the tree X^* . More on self-similar actions in general can be found in [7]. For our purposes the following observations suffice.

The action of a , b and c on X^* given by (1) can be rewritten in a recursive form as follows. For any word u over X ,

$$\begin{aligned} a(0u) &= 1u, & b(0u) &= 2u, & c(0u) &= 0c(u), \\ a(1u) &= 0u, & b(1u) &= 1b(u), & c(1u) &= 2u, \\ a(2u) &= 2a(u), & b(2u) &= 0u, & c(2u) &= 1u. \end{aligned} \quad (2)$$

This implies that, for any sequence g of moves, there exist a permutation π_g of X and three sequences of moves g_0, g_1 and g_2 such that, for every word u over X ,

$$g(0u) = \pi_g(0)g_0(u), \quad g(1u) = \pi_g(1)g_1(u), \quad g(2u) = \pi_g(2)g_2(u). \quad (3)$$

The permutation $\pi(g)$ is called the *root permutation* and it indicates the action of g on the first level of the tree (just below the root), while g_0, g_1 and g_2 are called the *sections* of g and indicate the action of g below the vertices on the first level. When (3) holds, we write

$$g = \pi_g (g_0, g_1, g_2)$$

and call the expression on the right a *decomposition* of g . Note that (3) may be correct for many different sequences of moves g_0 (or g_1 or g_2), but all these sequences represent the same element of the group H . Decompositions of the generators a, b and c are given by

$$a = (01) (1, 1, a), \quad b = (02) (1, b, 1), \quad c = (12) (c, 1, 1), \quad (4)$$

where 1 denotes the empty sequence of moves (the trivial automorphism of the tree). Two decompositions may be multiplied by using the formula (see [7] or [3])

$$gh = \pi_g (g_0, g_1, g_2) \pi_h (h_0, h_1, h_2) = \pi_{gh} (g_{h(0)}h_0, g_{h(1)}h_1, g_{h(2)}h_2). \quad (5)$$

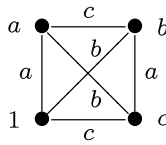


Fig. 7. The Schreier graph of A in H .

The decompositions of the generators a , b and c given in (4) and the decomposition product formula (5) are sufficient to calculate a decomposition for any sequence of moves. We refer to such calculations as decomposition calculations.

Theorem 6 (D'Angeli and Donno [1]). Two coupled configurations $U = \begin{pmatrix} u_T \\ u_B \end{pmatrix}$ and $V = \begin{pmatrix} v_T \\ v_B \end{pmatrix}$ on n disks are compatible if and only if the length of the longest common prefix of u_T and u_B is the same as the length of the longest common prefix of v_T and v_B .

Remark 4. Note that Theorem 6 implies that the $n + 1$ sets $C\Gamma_{n,0}, C\Gamma_{n,1}, \dots, C\Gamma_{n,n}$, where $C\Gamma_{n,i}$ consists of the coupled configurations $\begin{pmatrix} u_T \\ u_B \end{pmatrix}$ such that the length of the longest common prefix of u_T and u_B is i , are the connected components of the coupled Hanoi graph $C\Gamma_n$. The largest of these sets is $C\Gamma_{n,0}$. It consists of $6 \cdot 9^{n-1}$ vertices, which are the basic coupled configurations (defined in the introduction). More generally, the set $C\Gamma_{n,i}$ has $3^i \cdot 6 \cdot 9^{n-1-i}$ vertices, for $i = 0, \dots, n-1$, and $C\Gamma_{n,n}$ has 3^n vertices (moreover, $C\Gamma_{n,n}$ is canonically isomorphic to Γ_n through the isomorphism $u \leftrightarrow \begin{pmatrix} u \\ u \end{pmatrix}$).

Since every tree automorphism preserves prefixes, the connected components of the coupled Hanoi graph must be subsets of the sets $C\Gamma_{n,i}$. Thus, only the other direction (showing that each of the sets $C\Gamma_{n,i}$ is connected) is interesting and needs to be proved.

Consider the subgroup $A = \langle cba, acb, bac \rangle \leq H$ (introduced in [2] and called Apollonian group, because its limit space is the Apollonian gasket). It is known that this subgroup has index 4 in H and $H/A = C_2 \times C_2$ (where C_2 is cyclic of order 2). A sequence of moves g belongs to A if and only if the parities of the number of occurrences of the moves a , b and c in g are all odd or all even. The elements $1, a, b, c$ form a transversal for A in H . The Schreier graph of the subgroup A in H is given in Fig. 7. The vertices are denoted by the coset representatives (for instance, the vertex b is the coset bA).

Lemma 2. The Apollonian subgroup acts transitively on every level of the tree X^* .

Proof. The claim follows from the fact that H acts transitively on every level of the tree and that, for every generator s in $\{a, b, c\}$, there is a loop labeled by s in the Hanoi graph Γ_n .

Indeed, if $g(u) = v$, for some sequence of moves g , and g is in, say, the coset aA , then $g'g(u) = v$ and $g'g$ is in A , where $g' = h^{-1}ah$ and h is any sequence of moves from v to the vertex 2^n (note that $g' \in aA$ and $g'(v) = h^{-1}ah(v) = h^{-1}a(2^n) = h^{-1}(2^n) = v$). \square

Remark 5. A small modification of the above argument (using the corner loops to modify the parity of the number of occurrences of any generator) shows that the commutator subgroup H' also acts transitively on every level of the tree. The fact that H' acts transitively was proved in a different way by D'Angeli and Donno and used in their proof of Theorem 6. We provide a different proof of Theorem 6, based on the transitivity of the action of A , enabling us to provide good estimates in the General Problem for basic coupled configurations.

Lemma 3. The set $C\Gamma_{n,0}$ of basic coupled configurations on n disks is connected.

Proof. Let $\begin{pmatrix} u_T \\ u_B \end{pmatrix}$ and $\begin{pmatrix} v_T \\ v_B \end{pmatrix}$ be coupled configurations in $C\Gamma_{n,0}$.

Since H acts transitively on every level of the tree, there exists a sequence of moves h such that $h(u_T) = v_T$. Let $h(u_B) = v'_B$.

Without loss of generality, assume that the top configuration v_T starts by 2, while the bottom configuration v_B starts by 0. The configuration v'_B may start by either 0 or 1. If it starts by 1, a single

application of the sequence of 3 moves

$$cab = (01) (a, cb, 1),$$

does not affect v_T (note the trivial section at 2), and changes the first letter in the bottom configuration to 0. Thus, we may assume that both v'_B and v_B start by 0.

We are interested in sequences of moves g that do not affect any configurations that start by 2 (and thus do not affect v_T) and keep the first letter in the bottom configuration equal to 0. In other words, we are interested in sequences of moves that decompose as

$$g = (g_0, *, 1),$$

where $*$ represents the section at 1, in which we are not interested.

Three such sequences are (this can be verified by direct decomposition calculations)

$$cabcab = (cba, *, 1)$$

$$bacacaba = (acb, *, 1),$$

$$bcbcacac = (bac, *, 1).$$

Since $\langle cba, acb, bac \rangle = A$, these three decompositions imply that, for every sequence of moves g_0 in A , there exists a sequence of moves g in H , and in fact in A , such that

$$g = (g_0, *, 1).$$

Let $v'_B = 0v'$ and $v_B = 0v$. Since A acts transitively on each level of the tree, there exists g_0 in A such that $g_0(v') = v$. Therefore, there exists g in A such that $g(v'_B) = v_B$ and $g(v_T) = v_T$, completing the proof that $C\Gamma_{n,0}$ is connected. \square

The rest of the proof of [Theorem 6](#) follows, essentially, the same steps as the original proof of D'Angeli and Donno and, being short, is included for completeness. Indeed, once it is known that the largest sets $C\Gamma_{n,0}$ are connected, it is sufficient to observe that H is a self-replicating group.

Lemma 4. *Hanoi Towers group H is a self-replicating group of tree automorphisms, i.e., for every word u over X and every sequence of moves g in H , there exists a sequence of moves h in H such that, for every word w over X ,*

$$h(uw) = ug(w).$$

Proof. Let w be any word over X . Since

$$a(2w) = 2a(w), \quad cbc = 2b(w), \quad bcb(2w) = 2c(w),$$

it is clear that, for every sequence of moves g , there exists a sequence of moves h such that $h(2w) = 2g(w)$. By symmetry, for every letter x in X and every sequence of moves g , there exists a sequence of moves h such that

$$h(xw) = xg(w)$$

and the claim easily extends to words over X (and not just letters). \square

Proof (Proof of [Theorem 6](#)). Let u and u' be words of length i and $\begin{pmatrix} uw_T \\ uw_B \end{pmatrix}$ and $\begin{pmatrix} u'w'_T \\ u'w'_B \end{pmatrix}$ be two coupled configurations in $C\Gamma_{n,i}$. Since H acts transitively on the levels of the tree, there exists a sequence of moves h' in H such that $h' \begin{pmatrix} uw_T \\ uw_B \end{pmatrix} = \begin{pmatrix} u'w''_T \\ u'w''_B \end{pmatrix}$, for some w''_T and w''_B (in fact, one may easily find such h' for which $w''_T = w_T$ and $w''_B = w_B$, but this does not matter). Since $C\Gamma_{n-i,0}$ is connected, there exists a sequence of moves g such that $g \begin{pmatrix} w''_T \\ w''_B \end{pmatrix} = \begin{pmatrix} w'_T \\ w'_B \end{pmatrix}$. By the self-replicating property of H , there exists a sequence of moves h in H such that

$$hh' \begin{pmatrix} uw_T \\ uw_B \end{pmatrix} = h \begin{pmatrix} u'w''_T \\ u'w''_B \end{pmatrix} = \begin{pmatrix} u'g(w''_T) \\ u'g(w''_B) \end{pmatrix} = \begin{pmatrix} u'w'_T \\ u'w'_B \end{pmatrix}. \quad \square$$

Theorem 7 (General Problem for Basic Configurations). *The diameter $D(n)$ of the largest component $C\Gamma_{n,0}$ of the coupled Hanoi graph $C\Gamma_n$ (on n disks) satisfies, for $n \geq 3$, the inequalities*

$$2 \times 2^n \leq D(n) \leq 3.66 \times 2^n.$$

Proof. We follow the proof of Lemma 3, but keep track of the lengths of the sequences of moves involved and, when we have a choice (and know how to make it), try to use short sequences.

Let $U = \begin{pmatrix} u_T \\ u_B \end{pmatrix}$ and $V = \begin{pmatrix} v_T \\ v_B \end{pmatrix}$ be coupled configurations in the largest component $C\Gamma_{n,0}$ of the coupled Hanoi graph. Without loss of generality, assume that the top configuration v_T starts by 2, while the bottom configuration v_B starts by 0.

There exists a sequence of moves h of length at most $2^n + 2$ such that $h(u_T) = v_T$ and $h(u_B) = v'_B$, for some configuration v'_B that starts by 0. Indeed, at most $2^n - 1$ steps are needed to change the top configuration from u_T to v_T , and then at most three more steps (recall that $cab = (01)(a, cb, 1)$) are needed to make sure that the bottom configuration starts by 0.

Let $v'_B = 0v'$ and $v_B = 0v$. We claim that there exists a sequence of moves g_0 in A such that $g_0(v') = v$ and the number of moves in the sequence g_0 is no greater than $2^n - 1$. Indeed, if the shortest sequence of moves g_s between v' and v happens to be in A we may set $g_0 = g_s$ (note that v' and v are vertices in the Hanoi graph Γ_{n-1} of diameter $2^{n-1} - 1$). If g_s happens to be, say, in the coset aA , we may set $g_0 = g^{(2)}ag^{(1)}$, where $g^{(1)}$ is the shortest sequence of moves from v' to 2^{n-1} and $g^{(2)}$ is the shortest sequence of moves from 2^{n-1} to v . The length of the sequence $g_0 = g^{(2)}ag^{(1)}$ is no greater than $2(2^{n-1} - 1) + 1 = 2^n - 1$. Since the sequence of moves $g_s^{-1}g^{(2)}g^{(1)}$ represents a closed path in the graph Γ_{n-1} that does not go through any of the corner loops and since all cycles in Γ_{n-1} other than the three corner loops are labeled by elements in A , the sequence $g_s^{-1}g^{(2)}g^{(1)}$ is in A . Therefore

$$g_0A = g^{(2)}ag^{(1)}A = ag^{(2)}g^{(1)}A = ag_sA = aaA = A,$$

which is what we needed.

Direct decomposition calculations give

$$bab(cba)^2bab = (acaba, *, 1)$$

$$abc(acb)^2cba = (babcb, *, 1),$$

$$cb(cba)^2bc = (cbcac, *, 1),$$

and therefore, for any $k \geq 0$,

$$bab(cba)^{2k+2}bab = (a(cab)^{k+1}a, *, 1),$$

$$abc(acb)^{2k+2}cba = (b(abc)^{k+1}b, *, 1), \quad (6)$$

$$cb(cba)^{2k+2}bc = (c(bca)^{k+1}c, *, 1).$$

This calculation justifies the entries in the top three rows of Table 2. In this table, f is a sequence of moves and f_0 is the corresponding section at 0. The first letter of any word is fixed by f and the section at 2 is trivial. In other words, f decomposes as

$$f = (f_0, *, 1).$$

The lengths of the sequences f and f_0 , as written, are $\ell(f)$ and $\ell(f_0)$, and the ratio in the last column is the ratio $\ell(f)/\ell(f_0)$ (in the rows that depend on k , the ratio is the maximum possible ratio, taken for $k \geq 0$ and rounded up).

The entries in the remaining rows in Table 2 are easy to verify. For instance, for case $c \leftarrow a$, by direct decomposition calculation,

$$cabcab = (cba, *, 1) \quad (7)$$

Table 2Sequences of moves fixing v_T and moving v'_B .

Case	f	f_0	$\ell(f)$	$\ell(f_0)$	Ratio
$a \xleftarrow{-} a$	$bab(cba)^{2k+2}bab$	$a(cab)^{k+1}a$	$6k + 12$	$3k + 5$	2.4
$b \xleftarrow{-} b$	$abc(acb)^{2k+2}cba$	$b(abc)^{k+1}b$	$6k + 12$	$3k + 5$	2.4
$c \xleftarrow{-} c$	$cb(cba)^{2k+2}bc$	$c(bca)^{k+1}c$	$6k + 10$	$3k + 5$	2
$c \xleftarrow{-} a$	$cabcab$	cba	6	3	2
	$cabcab(cba)^{2k+2}bab$	$cb(cab)^{k+1}a$	$6k + 14$	$3k + 6$	2.34
$a \xleftarrow{-} b$	$bacacaba$	acb	8	3	2.67
	$bacacac(ach)^{2k+2}cba$	$ac(abc)^{k+1}b$	$6k + 16$	$3k + 6$	2.67
$b \xleftarrow{-} c$	$bcbcacac$	bac	8	3	2.67
	$bcbcacbcb(cba)^{2k+1}bc$	$ba(bca)^{k+1}c$	$6k + 14$	$3k + 6$	2.34
$b \xleftarrow{-} a$	$bcbcbcab$	$baba$	10	4	2.5
	$bcbcbcbcb(cba)^{2k+2}bab$	$bab(cab)^{k+1}a$	$6k + 18$	$3k + 7$	2.58
$c \xleftarrow{-} b$	$cabcaba$	$cbcb$	8	4	2
	$cabcac(ach)^{2k+2}cba$	$cbc(abc)^{k+1}b$	$6k + 16$	$3k + 7$	2.29
$a \xleftarrow{-} c$	$babcbabc$	$acac$	8	4	2
	$bab(cba)^{2k+3}bc$	$aca(bca)^{k+1}c$	$6k + 14$	$3k + 7$	2

and the entry in the next row is obtained simply by multiplying the equality (7) and the first equality in (6)

$$\begin{aligned}
 cabcb(cba)^{2k+2}bab &= (cab)(abba)(b(cba)^{2k+2}bab) \\
 &= (cabcb)(bab(cba)^{2k+2}bab) \\
 &= (cba, *, 1)(a(cab)^{k+1}a, *, 1) = (cbaa(cab)^{k+1}a, *, 1) \\
 &= (cb(cab)^{k+1}a, *, 1).
 \end{aligned}$$

All other cases are equally easy to verify (by verifying directly the basic case, and then multiplying it by a corresponding equality from (6) to obtain the cases depending on k).

Consider g_0 as defined above. There is no occurrence of aa , bb or cc in this sequence (since we always chose the shortest paths as we built g_0) and it is in A . The sequence g_0 is a product of factors each of which has the form of one of the entries in column f_0 in Table 2 or their inverses. Moreover, the decomposition is such that the length of g_0 is the sum of the lengths of the factors. Indeed, the entries in column f_0 and their inverses are all possible sequences of moves in A without occurrence of aa , bb or cc for which no proper suffix is in A . Such sequences correspond precisely to paths without backtracking in the Schreier graph in Fig. 7 that start at 1, end at 1 and do not visit the vertex 1 except at the very beginning and at the very end. There are 18 such types of paths, three choices for the first step (a , b or c) to leave vertex 1, three choices for the last step (a , b or c) to go back to vertex 1, and two choices for the orientation (order) used to loop around the three vertices (cosets) a , b and c before the return to 1 (positive or negative orientation). The column f_0 in the table only lists the 9 possible paths with negative orientation (and classifies the 9 cases by the first and last move), since the other 9 are just inverses of the entries in the table. For instance, the notation $c \xleftarrow{-} a$ indicates paths (sequences of moves) that start by the move a and end by the move c .

Once g_0 is appropriately factored, Table 2 can be used to define g of length no greater than $2.66\ell(g_0) \leq 2.66(2^n - 1)$ such that $g \begin{pmatrix} v_T \\ v'_B \end{pmatrix} = \begin{pmatrix} v_T \\ v_B \end{pmatrix}$.

Thus, we may arrive from the initial coupled configuration $\begin{pmatrix} u_T \\ u_B \end{pmatrix}$ to the final coupled configuration $\begin{pmatrix} v_T \\ v_B \end{pmatrix}$ in no more than $(2^n + 2) + 2.66(2^n - 1) \leq 3.66 \times 2^n$ moves. \square

It is evident that good understanding of the structure of $C\Gamma_{n,0}$, for all n , provides good understanding of $C\Gamma_{n,i}$, for all n and i . For instance, the understanding of the graphs $C\Gamma_{1,0}$ (6 vertices, diameter 2) and $C\Gamma_{2,0}$ (54 vertices, diameter 6) enabled the author to determine the exact values of the diameter of the two smallest nontrivial components $C\Gamma_{n,n-1}$ and $C\Gamma_{n,n-2}$, for any number of disks. For instance, the diameter of $C\Gamma_{n,n-1}$ is, for $n \geq 1$, equal to

$$\frac{7}{6}2^n - \frac{3 + (-1)^n}{6}.$$

The details will appear in a future work.

Acknowledgments

The author is grateful to Tullio Ceccherini-Silberstein and Alfredo Donno for their help, useful remarks, and corrections.

This material is based upon work supported by the National Science Foundation.

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